Group Averaging and Refined Algebraic Quantization

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We review the framework of Refined Algebraic Quantization and the method of Group Averaging for quantizing systems with first-class constraints. Aspects and results concerning the generality, limitations, and uniqueness of these methods are discussed.

1. Introduction

Refined Algebraic Quantization (RAQ) is an attempt (amongst others) to concretize Dirac's program for the quantization of constrained systems within a generally applicable, well defined mathematical framework. It was first formulated as a general scheme in [1,9] and recently developed further in [3,4]. Here I wish to report on these recent developments.

The method itself has already been used (partly implicitly) earlier in some successful applications to quantum gravity in specialized situations, like linearized gravity on symmetric backgrounds [6,7] or various minisuperspace models [10,11]. These suggested the program to find a general scheme of which these cases are just special cases.

Any scheme that deals with constrained systems needs to interpret the phrase 'solving the constraints'. Here the guiding idea of RAQ is to work from the onset within an auxiliary Hilbert space, $\mathcal{H}_{\text{aux}}$, by means of which a $\ast$-algebra of observables, $\mathcal{A}_{\text{obs}}$, is constructed *before* the constraints are 'solved'. The $\ast$-operation on $\mathcal{A}_{\text{obs}}$ derives from the adjoint-operation $\dagger$ on $\mathcal{H}_{\text{aux}}$, which allows to connect the auxiliary inner product with the inner product on the physical Hilbert space, $\mathcal{H}_{\text{phys}}$, since the latter is required to support a $\ast$-representation of $\mathcal{A}_{\text{obs}}$. A possibly non-trivial limitation derives from the fact that the constraint operators on $\mathcal{H}_{\text{aux}}$ are required to be self-adjoint, which may not be possible in case they do not form a Lie-algebra (i.e. 'close' with structure-functions only). This difficulty clearly does not arise if the constraints derive from the action of a Lie-group G. In the following we shall restrict attention to such cases. *More precisely, we consider situations where a finite dimensional Lie group G acts by some unitary representation U on $\mathcal{H}_{\text{aux}}$.*

2. Concretizing the Dirac Procedure

From the RAQ point of view, the concretization of the Dirac procedure starts form an auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ with inner product $\langle \cdot | \cdot \rangle_{\text{aux}}$ and the set of unitary operators $\{U(g)\, g \in G\}$. The naive reading of Dirac's prescription is to identify the space of physical states with those elements in $\mathcal{H}_{\text{aux}}$ that are fixed by G:

$$U(g)|\psi\rangle = |\psi\rangle, \quad \forall g \in G, \tag{1}$$

which just says that one should pick the trivial subrepresentation of U. But clearly this statement needs not be well defined since U might simply not contain the trivial representation as *sub*-representation. This will happen if the operators $U(g)$ do not all have the value 'one' in the discrete part of their spectrum. Another difficulty comes in if G is not unimodular. Then it has been convincingly argued using methods of geometric quantization (see [2] and references therein) that (1) is simply not the right condition, but that (1) should *formally* be replaced with

$$U(g)|\psi\rangle = \Delta^{1/2}(g)|\psi\rangle, \tag{2}$$

where $\Delta(g) := \det[\text{Ad}_{\text{g}}]$ is the modular function on G (a one-dimensional, real representation of G). We said 'formally' since (2) cannot hold in $\mathcal{H}_{\text{aux}}$, for this would mean that the unitary operator $U(g)$ can change lengths by $\Delta^{1/2}(g) \neq 1$,

a plain contradiction. This is connected with the first problem since it means that $\Delta^{1/2}(g)$ cannot be a discrete spectral value of $U(g)$. In physical terms, $|\psi\rangle$ in (2) is not normalizable. Hence one must read these equations in the appropriate sense. It is useful to break up the further development into several steps.

Step 1: We denote by $\dagger$ the adjoint map on operators on $\mathcal{H}_{\mathrm{aux}}$ with respect to $\langle\cdot|\cdot\rangle_{\mathrm{aux}}$. Choose a dense linear subspace $\Phi \subseteq \mathcal{H}_{\mathrm{aux}}$ which is left invariant (as set) by G's action. This choice of Φ is an important step and will generally require some physical input. Let Φ^* be the algebraic dual (linear functionals) of Φ. Put on Φ^* the topology of pointwise convergence: $f_n \to f$ in Φ^* iff $f_n[\phi] \to f[\phi]$ as real numbers for all $\phi \in \Phi$. Note that, conversely, each $\phi \in \Phi$ defines a *continuous* linear functional on Φ^* by setting $\phi(f) := f[\phi]$. Hence Φ embeds in the topological dual (continuous linear functionals) of Φ^*.

Step 2: Consider the set $\mathcal{L} := \{A \mid \mathcal{H}_{\mathrm{aux}} \supseteq D(A) \to \mathcal{H}_{\mathrm{aux}}\}$ of linear operators, where $D(A)$ denotes the domain of A. It is not an algebra due to mismatches of ranges and domains. We define a subset of $\mathcal{L}$ by

$$\mathcal{A} := \{A \in \mathcal{L} \mid \Phi \subseteq D(A) \cap D(A^\dagger)$$
$$\Phi \supseteq A(\Phi),\ \Phi \supseteq A^\dagger(\Phi)\}, \qquad (3)$$

and make it into an algebra by restricting the action of each $A \in \mathcal{A}$ to Φ. Since $\dagger$ also restricts to $\mathcal{A}$, we have a $*$-algebra which – without indicating the restriction to Φ – we continue to call $\mathcal{A}$. Note that, by definition of Φ, $\mathcal{A}$ contains the operators $U(g)$. *From now on, the $*$-operation of this algebra is the only trace left by $\langle\cdot|\cdot\rangle_{\mathrm{aux}}$.* Finally, we define a sub-$*$-algebra $\mathcal{A}_{\mathrm{obs}} \subseteq \mathcal{A}$ as the 'commutant' in $\mathcal{A}$ of the set $\{U(g)\ g \in G\}$:

$$\mathcal{A}_{\mathrm{obs}} := \{A \in \mathcal{A} \mid U(g)A = AU(g),\ \forall g \in G\}. \quad (4)$$

Step 3: Each $A \in \mathcal{A}$ acts as continuous linear map on Φ^* via the 'adjoint' action:

$$Af := f \circ A^*. \qquad (5)$$

Hence the constraint operators also act on Φ^* and we can define the *solution set*, $\mathcal{V}$, by

$$\mathcal{V} := \{f \in \Phi^* \mid U(g)f = \Delta^{1/2}(g)f\}, \qquad (6)$$

which, by construction, carries an anti-$*$-representation of $\mathcal{A}_{\mathrm{obs}}$ by continuous maps. (The 'anti' is a consequence of the $*$ in (5) and does no harm)

Step 4: The Hilbert space of physical states is now to be found within $\mathcal{V}$. Hence we seek a subspace $\mathcal{H}_{\mathrm{phys}} \subseteq \mathcal{V}$ with inner product $\langle\cdot|\cdot\rangle_{\mathrm{phys}}$ that makes $\mathcal{H}_{\mathrm{phys}}$ into a Hilbert space. Generally one cannot turn all of $\mathcal{V}$ into a Hilbert space, since this would imply that all $A \in \mathcal{A}_{\mathrm{obs}}$ were defined everywhere on $\mathcal{H}_{\mathrm{phys}}$ and hence bounded, which is too restrictive. (Note: Merely having the whole Hilbert space as domain does not yet imply boundedness of a linear operator A. But if A *and its adjoint* are defined everywhere boundedness follows.) Hence $\mathcal{H}_{\mathrm{phys}}$ will in general be a proper subset of $\mathcal{V}$ and the topology of $\mathcal{H}_{\mathrm{phys}}$ induced by $\langle\cdot|\cdot\rangle_{\mathrm{phys}}$ must be finer than that it inherits from Φ^*, since the former must be closed. This also means that operators in $\mathcal{A}_{\mathrm{obs}}$ will not necessarily be bounded on $\mathcal{H}_{\mathrm{phys}}$ and hence we do not have an anti-$*$-representation of $\mathcal{A}_{\mathrm{obs}}$ on $\mathcal{H}_{\mathrm{phys}}$ since domains and ranges might not match. Hence we proceed as usual by assuming that there is a dense subspace $\Phi_{\mathrm{phys}} \subseteq \mathcal{H}_{\mathrm{phys}}$ on which we have such a representation. But besides these technicalities the relevant condition on $\langle\cdot|\cdot\rangle_{\mathrm{phys}}$ is this: *the physical inner product is to be chosen such that for all $A \in \mathcal{A}_{\mathrm{obs}}$ we have $A^* = A^\dagger$ on A^*'s domain in $\mathcal{H}_{\mathrm{phys}}$, where $\dagger$ now denotes the adjoint operation with respect to $\langle\cdot|\cdot\rangle_{\mathrm{phys}}$. This is how $\langle\cdot|\cdot\rangle_{\mathrm{phys}}$ is influenced (but generally not determined) by $\langle\cdot|\cdot\rangle_{\mathrm{aux}}$. It is with respect to this adjoint operation that we speak of an anti-$*$-representation of $\mathcal{A}_{\mathrm{obs}}$ on Φ_{phys}.*

3. RAQ and the η-Map

RAQ concretizes step 4 of the Dirac procedure just outlined. It aims to construct $\mathcal{H}_{\mathrm{phys}}$ by finding a so-called η-map (or 'rigging map'), which is an antilinear map $\eta : \Phi \to \Phi^*$ such that the image of η consists entirely of solutions, i.e., $\eta(\Phi) \subseteq \mathcal{V}$. Further conditions on this map are

1. η is real: $\eta(\phi_1)[\phi_2] = \overline{\eta(\phi_2)[\phi_1]}$
2. η is positive: $\eta(\phi)[\phi] \geq 0$
3. η's image is an invariant domain for $\mathcal{A}_{\mathrm{obs}}$

and η intertwines the representations of $\mathcal{A}_{\text{obs}}$ on Φ and Φ^*:

$$A\eta(\phi) = \eta(A\phi) \tag{7}$$

Given such an η, one defines $\langle \cdot | \cdot \rangle_{\text{phys}}$ on its image by

$$\langle \eta(\phi_1) | \eta(\phi_2) \rangle_{\text{phys}} := \eta(\phi_2)[\phi_1] \tag{8}$$

and then defines $\mathcal{H}_{\text{phys}}$ as the completion of η's image with respect to the uniform structure (Cauchy sequences) defined by this inner product. *By construction, $\langle \cdot | \cdot \rangle_{\text{phys}}$ satisfies the condition that the $*$-operation on $\mathcal{A}_{\text{obs}}$ is the adjoint with respect to $\langle \cdot | \cdot \rangle_{\text{phys}}$.*

We need only check two points in order to see that the $\mathcal{H}_{\text{phys}}$ so defined satisfies the conditions of step 4: First, we wanted $\mathcal{H}_{\text{phys}}$ to be a subset of $\mathcal{V}$, so we need to check that the completion just mentioned does not add points outside $\mathcal{V}$, i.e., that $\mathcal{H}_{\text{phys}}$ is indeed a subset of $\mathcal{V}$ with finer intrinsic topology. To see this, we consider the map $\sigma : \mathcal{H}_{\text{phys}} \to \Phi^*$, defined by

$$\sigma(f)[\phi] := \langle \eta(\phi) | f \rangle_{\text{phys}} , \tag{9}$$

and note immediately that $\sigma(f)$ vanishes iff f is orthogonal to all elements in the images of η. But since the image is dense in $\mathcal{H}_{\text{phys}}$ by construction f must itself vanish, hence σ is injective. Next we prove that σ is continuous: if $f_n \to f$ in $\mathcal{H}_{\text{phys}}$ then $\langle \eta(\phi) | f_n \rangle_{\text{phys}} \to \langle \eta(\phi) | f \rangle_{\text{phys}}$ for all ϕ, since $\langle | \eta(\phi) | \cdot \rangle_{\text{phys}}$ is a continuous linear form on $\mathcal{H}_{\text{phys}}$. Hence $\sigma(f_n)[\phi] \to \sigma(f)[\phi]$ for all ϕ which, since Φ^* carries the topology of pointwise convergence, implies $\sigma(f_n) \to \sigma(f)$ and therefore continuity of σ.

The second point to check is that we indeed have an anti-$*$-representation of $\mathcal{A}_{\text{obs}}$ on a dense subspace Φ_{phys} in $\mathcal{H}_{\text{phys}}$. To show this, we simply identify Φ_{phys} with the image of η and note that by (5) and (7) we have for all $\phi \in \Phi$:

$$
\begin{aligned}
\sigma(Af)[\phi] &= \langle \eta(\phi) | Af \rangle_{\text{phys}} = \langle \eta(A^*\phi) | f \rangle_{\text{phys}} \\
&= \sigma(f)[A^*\phi] = A\sigma(f)[\phi].
\end{aligned}
\tag{10}
$$

Hence $\sigma : \Phi_{\text{phys}} \to \text{Image}(\sigma) \subseteq \Phi^*$ is an isomorphism of anti-$*$-representations of $\mathcal{A}_{\text{obs}}$.

An interesting question is how general this method of constructing $\mathcal{H}_{\text{phys}}$ via an η-map actually is; that is, given $\mathcal{H}_{\text{phys}}$ as in step 4 above,

is there always an η map whose image is $\mathcal{H}_{\text{phys}}$? Well, remember that each $\phi \in \Phi$ defines a continuous linear functional on Φ^*. Restriction to the linear subspace $\mathcal{H}_{\text{phys}}$ of Φ^* yields a continuous linear functional on $\mathcal{H}_{\text{phys}}$ with respect to its intrinsic topology, since the latter is finer than that induced by Φ^*. Hence, by Riesz' theorem, there is a a unique vector $\eta'(\phi) \in \mathcal{H}_{\text{phys}}$ which satisfies

$$\phi(f) := f[\phi] = \langle \eta'(\phi) | f \rangle_{\text{phys}} \tag{11}$$

for each $f \in \mathcal{H}_{\text{phys}}$. The map $\eta' : \Phi \to \mathcal{H}_{\text{phys}}$ is obviously antilinear and has a non-trivial image, since $\text{kernel}(\eta') = \bigcap_{f \in \mathcal{V}} \text{kernel}(f) \neq \Phi$ for $\mathcal{V} \neq \{0\}$. We can now define Φ'_{phys} to be the image of this η' and $\mathcal{H}'_{\text{phys}} \subseteq \mathcal{H}_{\text{phys}}$ its completion. This *almost* proves that we can at least reproduce the subspace $\mathcal{H}'_{\text{phys}}$ by an η-map if η' were indeed an η-map in the technical sense. But it might fail condition 3 above that it intertwines with $\mathcal{A}_{\text{obs}}$; the reason being that so far nothing ensures Φ'_{phys} to be contained in Φ_{phys}. Hence operators in $\mathcal{A}_{\text{obs}}$ might not act on Φ'_{phys} at al! However, if $\Phi'_{\text{phys}} \subseteq \Phi_{\text{phys}}$ then η' is indeed an η-map and we can at least claim to be able to reconstruct *some* sector $\mathcal{H}'_{\text{phys}} \subseteq \mathcal{H}_{\text{phys}}$.

Given that, we may finally wonder what additional assumptions would guarantee that we can construct *all* of $\mathcal{H}_{\text{phys}}$ by RAQ. One such condition is the following [3]: *If $\mathcal{H}_{\text{phys}}$ constructed by the Dirac procedure decomposes into a direct sum of superselection sectors, then each sector can be separately constructed by the Dirac procedure.* If this is satisfied we argue as follows: suppose $\mathcal{H}'_{\text{phys}}$ is a maximal subsector of $\mathcal{H}_{\text{phys}}$ which can be constructed by RAQ. Then its orthogonal complement is also a sector which, by hypothesis, is separately constructible by the Dirac procedure. But then, again by hypothesis, we can reconstruct a subsector of this orthogonal complement, which contradicts the assumption that $\mathcal{H}'_{\text{phys}}$ was maximal.

4. Group Averaging

Whereas the η-map is a specific way to build $\mathcal{H}_{\text{phys}}$ for step 4 in the Dirac procedure, Group Averaging is in turn a specific way to construct an η-map. In Dirac's $\langle \text{bra} | \text{ket} \rangle$-terminology, the

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idea is simply to define (and make sense of)

$$\eta|\phi\rangle := \int_G d\mu(g)\,\langle\phi|U(g)\,, \qquad (12)$$

where $d\mu$ is some appropriately chosen measure on G. If G were unimodular, the right- and left-invariant Haar measures coincide and are the correct choice for $d\mu$ in order for η's image to formally solve (1) *and* make η real in the sense of condition 1 above. Note that this reality condition requires the measure $d\mu(g)$ to be invariant under the inversion map $I : g \to g^{-1}$. For non-unimodular groups neither the left- nor the right-invariant measure is I-invariant. Instead one has to take the 'symmetric' measure, defined by

$$d\mu_0(g): \quad = \quad \Delta^{1/2}(g)\,d\mu_L(g)$$
$$= \quad \Delta^{-1/2}(g)\,d\mu_R(g)\,, \qquad (13)$$

where $d\mu_L, d\mu_R$ are the left- and right-invariant measures respectively. Moreover, this measure also implies that η's image (formally) satisfies the modified equation (2). We refer to [4] for a more extensive discussion of the modified Dirac condition and how it can be derived by Group Averaging from an appropriate adaptation of the 'unimodularization' technique originally developed in geometric quantization [2].

The physical inner product according to Group Averaging is given by

$$\langle\eta(\phi_2)|\eta(\phi_1)\rangle_{\mathrm{phys}} := \int_G d\mu_0(g)\langle\phi_1|U(g)\phi_2\rangle_{\mathrm{aux}}(14)$$

so that the Group Averaging procedure only makes sense for states $\phi_{1,2}$ for which this integral converges absolutely. But it turns out [4] that a more restricted choice of Φ is appropriate: we say that ϕ is an L^1 state, iff

$$\int_g d\mu_n\langle\phi|U(g)\phi\rangle_{\mathrm{aux}} \qquad (15)$$

converges absolutely for all integers n, where $d\mu_n(g) := \Delta^{n/2}(g)d\mu_0$. One can then also construct the group algebra $\mathcal{A}_G$ of functions on G which are L^1 with respect to $d\mu_n$ for all n, and prove that its action on L^1 states results again in L^1 states . This fact allows to prove a uniqueness theorem in the following form (see [4] for more details):

Theorem 1 *Suppose* Φ *is an* L^1 *subspace of* $\mathcal{H}_{\mathrm{aux}}$ *which is invariant under* $\mathcal{A}_G$. *Then, up to overall scale, any* η-*map is of the form (12) with* $d\mu = d\mu_0$.

5. Summary

We outlined two technical devices which, to a certain extent, concretize the Dirac approach to quantizing constrained systems along the sequence [Group Averaging] $\to$ [η-map] $\to$ [Dirac approach]. There are fairly strong uniqueness theorems regarding these devices, provided Group Averaging converges sufficiently rapidly. But there is no general characterization when this can actually be achieved. Physical as well as mathematical inputs are required, perhaps most importantly in the choice of Φ, and results are likely to delicately depend on this choice. Recent investigations of convergence properties in contexts of specific models [5,8] confirm this general expectation.

Finally we mention the obvious, namely that the restriction to systems whose first class constraints close with structure *constants* (i.e. form a Lie algebra) should be lifted. After all, GR is not in this form. We plan to investigate this in the future.

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